Relationship between spread in ϕ and entanglement purity:

★ Consider a state that is $\frac{1}{\sqrt{2}}(|H\rangle|H\rangle+|V\rangle|V\rangle)$ with probability p
$\{50\%\ |H\rangle|H\rangle,\ 50\%\ |V\rangle|V\rangle\}$ with probability $(1-p)$.

The density operator for this state is

$$\rho_I = \sum_i p_i |\psi_i\rangle\langle\psi_i| = p\left[\frac{1}{\sqrt{2}}(|H\rangle|H\rangle+|V\rangle|V\rangle)(\langle V|\langle V|+\langle H|\langle H|)\frac{1}{\sqrt{2}}\right]$$
$$+\frac{1-p}{2}|H\rangle|H\rangle\langle H|\langle H| + \frac{1-p}{2}|V\rangle|V\rangle\langle V|\langle V|$$
$$=\frac{1}{2}|H\rangle|H\rangle\langle H|\langle H| + \frac{1}{2}|V\rangle|V\rangle\langle V|\langle V|$$
$$+\frac{p}{2}|H\rangle|H\rangle\langle V|\langle V| + \frac{p}{2}|V\rangle|V\rangle\langle H|\langle H|,$$

or, in the $\{|H\rangle|H\rangle, |H\rangle|V\rangle, |V\rangle|H\rangle, |V\rangle|V\rangle\}$ basis, the density matrix is

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p & 0 & 0 & 1 \end{bmatrix}.$$

★ Now consider instead a state that is $\frac{1}{\sqrt{2}}(|H\rangle|H\rangle+e^{i\phi}|V\rangle|V\rangle)=|\psi_\phi\rangle$ but with ϕ uniformly distributed over the interval $-\Delta$ to $+\Delta$.

The density operator for this state is

$$\rho_{II} = \sum_i p_i|\psi_i\rangle\langle\psi_i| \longrightarrow \int_{-\Delta}^{\Delta} d\phi\ \left(\frac{1}{2\Delta}\right)|\psi_\phi\rangle\langle\psi_\phi|$$
$$=\frac{1}{2\Delta}\int_{-\Delta}^{\Delta} d\phi\left[\frac{1}{\sqrt{2}}(|H\rangle|H\rangle+e^{i\phi}|V\rangle|V\rangle)(\langle V|\langle V|e^{-i\phi}+\langle H|\langle H|)\frac{1}{\sqrt{2}}\right]$$
$$=\frac{1}{2\Delta}\int_{-\Delta}^{\Delta} d\phi\left[\frac{1}{2}|H\rangle|H\rangle\langle H|\langle H| + \frac{1}{2}|V\rangle|V\rangle\langle V|\langle V| + \frac{e^{-i\phi}}{2}|H\rangle|H\rangle\langle V|\langle V| + \frac{e^{i\phi}}{2}|V\rangle|V\rangle\langle H|\langle H|\right]$$
$$=\frac{1}{2\Delta}\left[\frac{\cancel{2}\Delta}{\cancel{2}}|H\rangle|H\rangle\langle H|\langle H| + \frac{\cancel{2}\Delta}{\cancel{2}}|V\rangle|V\rangle\langle V|\langle V| + \frac{\cancel{2}\sin\Delta}{\cancel{2}}|H\rangle|H\rangle\langle V|\langle V| + \frac{\cancel{2}\sin\Delta}{\cancel{2}}|V\rangle|V\rangle\langle H|\langle H|\right]$$

so $\rho_{II} = \frac{1}{2}|H\rangle|H\rangle\langle H|\langle H| + \frac{1}{2}|V\rangle|V\rangle\langle V|\langle V|$

$$+ \frac{\left(\frac{\sin\Delta}{\Delta}\right)}{2}|H\rangle|H\rangle\langle V|\langle V| + \frac{\left(\frac{\sin\Delta}{\Delta}\right)}{2}|V\rangle|V\rangle\langle H|\langle H| \,,$$

or, in the $\{|H\rangle|H\rangle, |V\rangle|V\rangle, |V\rangle|H\rangle, |V\rangle|V\rangle\}$ basis,
the density matrix is
$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \frac{\sin\Delta}{\Delta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\sin\Delta}{\Delta} & 0 & 0 & 1 \end{bmatrix}.$$

We can see that the two ways of describing the overall state are equivalent as long as $\frac{\sin\Delta}{\Delta} = p$.

Reality checks: for $\Delta = 0$, $\frac{\sin\Delta}{\Delta} \to 1$ and $p = 1$ ✓. phase completely known ⇒ pure state.

for $\Delta = \pi$, $\frac{\sin\Delta}{\Delta} = 0$ and $p = 0$ ✓ phase completely unknown ⇒ no entanglement

This model assumes a flat probability distribution for ϕ over its range. This is likely unrealistic, but a similar argument applies if we let ϕ have any symmetric probability distribution around $\phi = 0$. We can complicate the model further by letting ϕ have an asymmetric distribution around its mean, but this is probably not important for conveying the physical idea of what is going on in this experiment!